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LETTER TO THE EDITOR

Landauer formula without Landauer's assumptions

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Abstract
The Landauer formula for dissipationless conductance lies at the heart of
modern electronic transport, yet it remains without a clear microscopic basis.
We analyse the Landauer formula microscopically and give a straightforward
quantum kinetic derivation for open systems. Some important experimental
implications follow. These lie beyond the Landauer result as popularly received.

In 1957 Rolf Landauer published a prescient interpretation of metallic resistivity [1]. It
heralded one of the most dramatic predictions of modern condensed-matter physics: the
perfect quantization, in steps of $2e^2/h$, of electrical conductance in one-dimensional metallic
channels [2]. Such quantization is quite independent of the material properties of the contact
and of its leads. It is universal insofar as one may validly neglect the disruptive influences of
inelastic dissipation within the transport process.

Landauer argued that the current, not the applied electromotive voltage, should be
understood as the active probe by which a device reveals its conductance. The observed carrier
flux is understood as a kind of diffusive flow, tending to shift carriers from a 'high'- to a 'low'-
density reservoir (lead). In the mesoscopic realm, this flow between leads is conditioned by the
intervening device channel, which presents a quantum tunnelling barrier to the non-interacting
electrons making up the flux.

The Landauer formula then has two cardinal tenets:

 (i) current is the flow of independent and degenerate electrons as they follow a nominal
     density gradient across reservoirs, and
 (ii) conductance is *lossless* transmission through an interposed quantum barrier.

These underpin Landauer's assumptions, namely that

 (a) transport ensues when a pair of leads connected to the device are set to different chemical
     potentials μ_L, μ_R;

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(b) the density mismatch due to $\mu_L - \mu_R$ sustains the current;

(c) $\mu_L - \mu_R$ is the applied voltage across the device;

(d) the Fermi energy is much larger than the thermal and electrical energies; and

(e) there are no inelastic processes to dissipate the electrical energy gained by the electrons.

Energy dissipation does not appear in the classic Landauer derivation [1, 3]. For a sample of mesoscopic dimensions, the model admits only elastic barrier scattering and excludes any role for inelastic processes within the active device and its interfaces. Yet it is dissipative inelastic scattering, and that alone, which ensures the energetic stability of resistive transport and hence a steady state for conduction.

Finite conductance and electrical energy loss are indivisible phenomena. The fundamental expression of their basic inseparability is the fluctuation-dissipation relation [4]. This establishes the equivalence of the mean-square fluctuation strength for the current and the conductance coefficient G in the energy dissipation rate $P = GV^2$, where V is the applied voltage.

There is a missing link between Landauer's universal—and lossless—conductance formula, which has been critical in the development of mesoscopic science [3], and the dissipative inelastic processes that are absolutely vital to the microscopic origin of resistance. Repeated attempts have been made to obtain the Landauer formula from microscopic-like arguments [2, 5, 6]; see also [7]. However, a convincing resolution has not yet materialized [8, 9]. The absence of so crucial a connection is a puzzling theoretical conundrum for Landauer's approach to mesoscopics, which is otherwise so empirically compelling.

In this letter we answer the question: how can the Landauer formula, in seemingly bypassing *all* inelastic processes, predict a finite—invariably dissipative—conductance that fulfils the fluctuation-dissipation theorem (FDT)? Below we offer a straightforward microscopic interpretation of Landauer's result for a mesoscopic contact open to the macroscopic environment.

Our treatment differs from all earlier attempts by directly addressing the essential physics of dissipation. To obtain conductance quantization within an open contact, the explicit interplay of elastic and dissipative processes is necessary *and sufficient*. Neglect of either mechanism, in favour of the other, negates the formula's microscopic basis. Both kinds of scattering are needed.

We also show that the traditional Landauer assumptions of pseudo-diffusive current and lossless scattering are not required in a first-principles analysis of Landauer conductance. Our model relies solely upon orthodox quantum kinetics, as embodied in the microscopic Kubo–Greenwood (KG) formalism [10, 11]. The KG formulation automatically guarantees the FDT; it is not invoked as an additional hypothesis. Both dissipative and lossless scattering appear within the resulting fluctuation-dissipation relation and both are assigned *equal* physical importance.

First, we briefly recall the KG formula and the essential charge conservation built into it. Next we discuss the form of the KG relaxation time, which fixes the conductance. Finally, we show how the physical constraints on a one-dimensional *open* ballistic channel, connected to macroscopic leads, lead naturally to Landauer's ideal quantized conductance. We go on to examine some of the measurable effects of device non-ideality on the Landauer conductance.

The KG theory [10, 11] describes the carriers' full many-body density matrix. All of the transport and fluctuation properties are contained within it. Thus, the conductivity for the system appears as the trace of the current–current correlation function:

$$\sigma(t) = \frac{ne^2}{m^*} \int_0^t \mathcal{C}_{vv}(t)\,\mathrm{d}t. \tag{1}$$

Here n is the carrier density and m^* is the effective mass. The velocity auto-correlation has the canonical form

$$C_{vv}(t) = \frac{\langle [v(t), v(0)] \rangle}{\langle v(0)^2 \rangle} \sim \exp(-t/\tau_m) \tag{2}$$

where the expectations trace over the equilibrium density matrix (this gives the leading, linear term in the expansion of the non-equilibrium response). For long times, the characteristic relaxation rate $1/\tau_m$ yields the dominant decay of the correlation.

The asymptotic relaxation rate subsumes, *on an equal footing*, the microscopic contribution from every physically relevant collision mechanism. Moving now to the long-time form of equation (1), the conductivity becomes

$$\sigma \to \frac{ne^2 \tau_m}{m^*}. \tag{3}$$

This is the celebrated Drude formula.

Equation (1) embodies the fluctuation-dissipation relation. In addition, its KG structure ensures that charge conservation is rigidly satisfied in the large, as well as locally [8]. This is an absolute prerequisite for open conductors as they exchange carriers freely with the outside.

Since Landauer's classic result applies to transport in a one-dimensional metallic wire, we examine equation (3) in one dimension (1D) for a single metallic sub-band (channel) within the wire. In the degenerate limit the density is $n = 2k_F/\pi$ in terms of the Fermi wavenumber k_F. The conductance over a sample of length L becomes

$$G \equiv \frac{\sigma}{L} = \frac{2k_F e^2}{\pi L m^*} \tau_m = \frac{2e^2}{h} \left(\frac{2\hbar k_F}{Lm^*} \tau_m \right) \equiv \frac{2e^2}{h} \mathcal{T}_{KG}, \tag{4}$$

in which the transmission coefficient $\mathcal{T}_{KG} = 2v_F \tau_m/L$ is proportional to the ratio of the overall scattering length, $v_F \tau_m$, to the operational length of the system.

Crucially, the many-body collisions (phonon emission, Coulomb scattering, etc) that redistribute the carriers' energy gain and cause dissipation are incorporated in τ_m alongside elastic impurity and barrier scattering. While elastic effects are explicitly invoked by the Landauer model, dissipative ones are neglected. It is dissipation that stabilizes the transport and substantiates the fluctuation-dissipation relation, equation (1).

Equation (4) is fully consistent with the Landauer formula, which is identical to it except that, in the accepted treatment, its transmission parameter $\mathcal{T}$ is ideal: $\mathcal{T} = 1$. In cases where $\mathcal{T}$ is *not* ideal the Landauer picture assumes that the non-ideality is due solely to elastic back-scattering from the barrier but does not facilitate the actual computation of $\mathcal{T}$. When inelastic scattering dominates, this picture is inapplicable [12].

Let us take a simple model for $\mathcal{T}_{KG}$. The wire is ballistic (impurity-free) and it is uniform; by Poisson's equation, so are the driving field and carrier distribution set up within it. At a distance L apart lie the wire–lead interfaces where the current is, in effect, injected and extracted by an outside generator. We observe that L is *not* a lithographically precise dimension. It characterizes the maximum physical scale for any collision process to occur in the entire mesoscopic assembly (the open wire, the interfaces and the reservoirs are one whole system). Note also that it is the external supply and removal of the current that explicitly energizes the open system [8]. There is no appeal in equation (1) to chemical-potential differences in any way, shape or form.

The wire–reservoir interfaces are zones of strong elastic scattering with impurities in the leads (the relaxation time is τ_{el}); equally, they are sites for strong *dissipative* interactions with the background modes excited by the influx and efflux of carriers from the current source

(the relaxation time is τ_{in}). The scattering mechanisms are stochastically independent, so that Matthiessen's rule applies:

$$\frac{1}{\tau_m} = \frac{1}{\tau_{\mathrm{el}}} + \frac{1}{\tau_{\mathrm{in}}}. \tag{5}$$

The mean free path (MFP) associated with the elastic collisions is obviously L, since by hypothesis that is the operational size of our impurity-free wire. Therefore $\tau_{\mathrm{el}} = L/v_{\mathrm{F}}$ for carriers at the Fermi level. By the same token, the MFP for inelastic scattering cannot be *greater* than L, though it may well be less at high currents[2]. Then

$$\tau_{\mathrm{in}} \leqslant \tau_{\mathrm{el}} = L/v_{\mathrm{F}}. \tag{6}$$

We conclude that

$$\mathcal{T}_{\mathrm{KG}} = \frac{2v_{\mathrm{F}}}{L}\left(\frac{\tau_{\mathrm{el}}\tau_{\mathrm{in}}}{\tau_{\mathrm{el}} + \tau_{\mathrm{in}}}\right) = \frac{2\tau_{\mathrm{in}}}{\tau_{\mathrm{in}} + L/v_{\mathrm{F}}} \leqslant 1. \tag{7}$$

It is the direct competition between the elastic processes in the mesoscopic system (as a ballistic structure, its elastic MFP is also its characteristic length) and the dissipative processes (ideally restricted to the current injection/extraction areas bounding L, but also liable to intrude into the interior) that determines the physical, and measurable, transmission through the sample.

What is the optimum outcome for equation (7) and what does it yield for the conductance? The maximum value of $\mathcal{T}_{\mathrm{KG}}$ is unity and it is attained precisely when

$$\tau_{\mathrm{in}} = \tau_{\mathrm{el}} = L/v_{\mathrm{F}}. \tag{8}$$

In other words, no inelastic events intrude into the core of the wire; they all occur at the interfaces. From equation (4) one easily discerns the corresponding value of G for this open, maximally ballistic 1D wire. It is nothing other than the Landauer conductance $G_0 = 2e^2/h$.

This establishes our key result. As with Landauer's derivation, we base it on two hypotheses:

 (i) that the wire is uniform, and
 (ii) that its 1D conduction sub-bands are well enough separated in energy that each can be treated independently.

Our account of the Landauer formula makes no use at all of the three other assumptions that are traditionally relied upon to establish the formula. They are:

- *that a mesoscopic current flows only when there is a density mismatch between carrier reservoirs, held at different chemical potentials*,
- *that coherent elastic scattering is the exclusive transmission mechanism mediating the conductance*,
- *that dissipation in an open conductor (accepted as vital in order to save the FDT) is a remote effect deep in the reservoirs, of no physical consequence for transport.*

We have demonstrated that these assumptions are superfluous in obtaining Landauer's result. A fourth key assumption remains:

- *that the quantized-conductance formula requires linear response in a degenerate channel.*

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[2] In a 'diffusive' wire, containing many elastic scattering centres, the complementary scenario holds: the wire length L represents a maximum scale for inelastic scattering, so that $\tau_{\mathrm{el}} < \tau_{\mathrm{in}} \leqslant L/v_{\mathrm{F}}$.

We now show that this hypothesis too is not required for understanding the microscopic basis of mesoscopic conductance.

A standard kinetic approach suffices to describe the carriers in a ballistic and uniform 1D conductor [13]. In steady state, with a driving field E (to be determined), our model carrier distribution function f_k in wavevector space $\{k\}$ obeys the transport equation

$$\frac{eE}{\hbar}\frac{\partial f_k}{\partial k} = -\frac{1}{\tau_{\text{in}}(\varepsilon_k)}\left(f_k - \frac{\langle\tau_{\text{in}}^{-1}f\rangle}{\langle\tau_{\text{in}}^{-1}f^{\text{eq}}\rangle}f_k^{\text{eq}}\right) - \frac{1}{\tau_{\text{el}}(\varepsilon_k)}\frac{f_k - f_{-k}}{2}. \tag{9}$$

The scattering times $\tau_{\text{in}}(\varepsilon_k)$ and $\tau_{\text{el}}(\varepsilon_k)$ are in general dependent on the band energy ε_k.

The properties of equation (9) impact directly upon the measurable transport behaviour. First, one and only one chemical potential μ enters the problem via the equilibrium Fermi–Dirac distribution at temperature T:

$$f_k^{\text{eq}} = 1/\{1 + \exp[(\varepsilon_k + \varepsilon_i - \mu)/k_{\text{B}}T]\}.$$

Quite generally, this is the reference state for computing the non-equilibrium function f_k [14] (here ε_i is the energy threshold of the sub-band). The applicability of equation (9) stretches over the entire range of density $n = \langle f\rangle$ from classical to strongly degenerate.

Second, the kinetic equation is microscopically conserving. On the right-hand side of equation (9), the leading, inelastic, collision term has a restoring contribution proportional to the expectation

$$\langle\tau_{\text{in}}^{-1}f\rangle = \int_{-\infty}^{\infty}\frac{2\text{d}k}{2\pi}\tau_{\text{in}}^{-1}(\varepsilon_k)f_k.$$

Finally, the second term on the right of equation (9) represents the elastic collisions, acting to restore symmetry to f_k. Both the elastic and inelastic terms satisfy gauge invariance.

The transport equation is analytically solvable when the collision times are independent of the electronic band energy [13]. At low currents, the solution has a transport behaviour identical to the KG formula described above. At high currents, for which neither the KG nor the Landauer expressions strictly apply, the kinetic solution remains tractable.

We now obtain G. As we have recalled, the common derivation of the Landauer conductance posits a highly degenerate electronic sub-band [3]. That is, we are in the zero-temperature limit. If the sub-band is populated even vestigially, the ideal conductance $G = G_0$ always emerges; but if the band is empty (the only other possibility at zero temperature), there is no transport and $G = 0$. There is no room for the intermediate band-threshold state that is expected at finite temperature.

The above approach cannot be used in a realistic setting, where the Fermi energy may well match the thermal energy. Experimentally [15, 16], the carrier density in a 1D channel is controlled via an adjacent gate. As the gate-bias voltage becomes more positive, the electron population undergoes a continuous change, from a low-density classical regime to a high-density degenerate one.

This classical-to-quantum transition is readily accommodated. Classically, the elastic MFP no longer scales with the Fermi velocity, but with the thermal velocity $v_{\text{th}} = \sqrt{2k_{\text{B}}T/m^*}$. In the general case, τ_{el} is given by the expression

$$\tau_{\text{el}}(n, T) = \frac{L}{\bar{v}(n, T)} \equiv L\frac{n}{\langle|v|f^{\text{eq}}\rangle}. \tag{10}$$

For a sparse, classical channel population, the characteristic mean velocity $\bar{v}(n, T)$ goes to v_{th}. For a dense, and thus degenerate, population, $\bar{v} = v_{\text{F}} = \sqrt{2(\mu - \varepsilon_i)/m^*}$, which holds
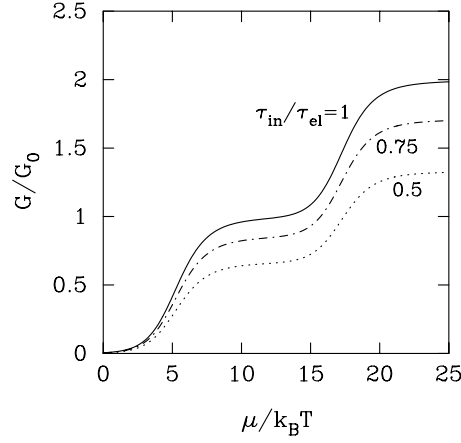
Figure 1. Conductance of a one-dimensional ballistic wire, computed with the kinetic model of equation (9). We show G scaled to the Landauer quantum G_0, as a function of chemical potential μ in units of thermal energy. G exhibits strong shoulders as μ successively crosses the sub-band-energy thresholds set at $\varepsilon_1 = 5\ k_B T$ and $\varepsilon_2 = 17\ k_B T$. Well above each threshold, sub-band electrons are strongly degenerate and the conductance tends to a well-defined quantized plateau; well below each threshold, the population and its contribution to G vanish as $\exp[-(\varepsilon_i - \mu)/k_B T]$. Full curve: G in a ballistic channel. This is the ideal limit for which the collision-time ratio τ_{in}/τ_{el} is unity. Chain curve: non-ideal case for $\tau_{in}/\tau_{el} = 0.75$. Note how the increased inelastic scattering brings down the plateaux. Dotted curve: the case of $\tau_{in}/\tau_{el} = 0.5$. The departure from ideality is now pronounced.

for equation (7) above. We can then extend equations (4) and (8) for G and $\mathcal{T}_{KG}$ to the whole regime of densities n_i in the ith sub-band accessible at finite temperature:

$$G_i = G_0 \left(\frac{h n_i}{2m^* \bar{v}(n_i, T)} \right) \left(1 - \frac{1}{1 + \tau_{in}/\tau_{el}(n_i, T)} \right), \qquad (11)$$

where v_F is replaced with its equivalent expression in 1D: $v_F = \hbar k_F/m^* = h n_i/4m^*$.

When the system is at low density ($\mu - \varepsilon_i \ll k_B T$) the conductance vanishes with n_i. When the system is degenerate ($\mu - \varepsilon_i \gg k_B T$) the conductance reaches a plateau, which is ideally quantized when $\tau_{in} = \tau_{el}$. In between, it rises smoothly as the chemical potential and density are systematically swept from far below the sub-band threshold ε_i to far above it.

The result is depicted in figure 1. We see there the total conductance of a 1D wire:

$$G = \sum_i G_i((\mu - \varepsilon_i)/k_B T),$$

made up of its individual sub-band contributions computed from equation (11), with full temperature dependence. The shoulders at the two sub-band thresholds are clear. In an idealized scenario (recall equation (8)), the characteristic Landauer plateaux appear as expected. As the inelastic scattering rate $1/\tau_{in}$ progressively exceeds the elastic rate $1/\tau_{el}$ (always keyed to the operational length of the structure), it is also evident that there is a progressive loss of ideality. Nonetheless the Landauer steps survive robustly, albeit at a reduced height commensurate with the degree of inelasticity.

To date, non-ideal behaviour in G has been viewed practically as an experimental nuisance, detracting from the aim of detecting the perfect Landauer prediction in ballistic wires [15, 16]. On the contrary, we suggest that the observed deviations from the ideal, for actual mesoscopic samples, carry valuable information on non-equilibrium transport effects. That these departures can, and should, be probed systematically follows from the logic of the

microscopic analysis presented above, supplemented with further detailed modelling of the collision terms entering into equations (1) and (9).

Our results, obtained from the standard KG theory and, equally well, from the solution of a standard kinetic equation, show how the Landauer conductance formula arises directly from a fine-scale interplay of elastic and inelastic processes in one-dimensional ballistic conductors. Such a derivation automatically respects charge conservation and the FDT. The latter is a natural outcome of the analysis, not an additional hypothesis to be imposed *ad hoc*.

We have shown that the Landauer theory's traditional phenomenological assumptions are not required for the validity of the formula itself, provided the essential physics of *resistive energy dissipation* is respected. Once the inelastic processes responsible for dissipation are properly included, the scope and value of the Landauer conductance formula extend well beyond Landauer's original conception. A minimal set of assumptions, as befits any microscopically based approach, is not only enough to recover the full Landauer formula; it also reveals considerably more information.

Finally, one conclusion stands out. In a mesoscopic ballistic conductor open to its electrical environment, the close interaction between dissipative and elastic scattering governs the behaviour of the conductance. It does so uniquely. Neither of the collision modes, acting alone, can sustain the physics of mesoscopic transport. A theory of transport must allow all such processes to act in concert, as they do in nature.

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